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Bernstein Type Theorems for Compact Sets in $\mathbf{R}^n$ Revisited

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In this paper we complete some results of (*J. Approx. Theory* **69** (1992), 156–166) and give a geometrical approach to the multivariate Bernstein and Markov inequalities. The most interesting and slightly surprising result is a sharp Markov inequality for convex symmetric subsets of $\mathbf{R}^n$ formulated in geometrical language. A sharp inequality for gradients of polynomials extends an old Kellog result (*Math. Z.* **27** (1927), 55–64), and it is also a partial positive answer to a question formulated by Wilhelmsen (*J. Approx. Theory* **11** (1974), 216–220) in 1974.

1. BERNSTEIN AND BERNSTEIN–MARKOV TYPE INEQUALITIES FOR A FAT COMPACT SET IN $\mathbf{R}^n$

Here we treat $\mathbf{R}^n$ as a subset of $\mathbf{C}^n$ such that $\mathbf{C}^n = \mathbf{R}^n + i\mathbf{R}^n$. Let $\Omega \subset \mathbf{R}^n$ and let U be a neighbourhood of the origin in $\mathbf{R}^n$. Given a function $f\colon \Omega + iU \to \mathbf{R}$, let us consider the Dini derivative in direction of a vector $v \in S^{n-1}$,

$$D_{v+} f(x) = \varliminf_{t \to 0+} \frac{1}{t} (f(x+itv) - f(x)),$$

for $x \in \Omega$. Note that for the usual gradient of a C^1 function $g\colon \mathbf{R}^n \supset \Omega \to \mathbf{R}$ (grad $g(x) = (D_1 g(x), ..., D_n g(x))$, where $D_i g = (\partial/\partial x_i) g$) we have

$$|\text{grad } g(x)| = \sup\{|D_v g(x)| : v \in S^{n-1}\}.$$

Here $D_v g(x)$ denotes the usual derivative in direction of a vector v. This formula is a consequence of the equalities

$$\bar{B}_n^* = \bar{B}_n$$

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and

$$\operatorname{extr} \bar{B}_n = S^{n-1}.$$

Here $B_n = \{|x| < 1\}$. (For the definition of E^*, see Section 2.) If E is a compact set in $\mathbf{C}^n$ we denote by V_E the generalized Green function of the open set $\mathbf{C}^n \setminus E$; see [S, BT]. Now we can state new version of the Theorem 1.14 of [B1].

1.1. THEOREM. *Let Ω be a bounded open set in $\mathbf{R}^n$ and $E = \bar{\Omega}$. Then for every $x \in \Omega$ and for a real polynomial p with* deg $p \leqslant k$ *we have the inequality*

$$|D_v p(x)| \leqslant k D_{v+} V_E(x)(\|p\|_E^2 - p^2(x))^{1/2}$$

for each $v \in S^{n-1}$.

The proof of Theorem 1.1 is in substance the same as that of Theorem 1.14 of [B1].

1.2. COROLLARY. *With the assumptions of Theorem* 1.1 *we have the inequality*

$$|\operatorname{grad} p(x)| \leqslant k \, |\operatorname{grad}_+ V_E(x)| \, (\|p\|_E^2 - p^2(x))^{1/2},$$

where

$$|\operatorname{grad}_+ V_E(x)| := \sup\{D_{v+} V_E(x) : v \in S^{n-1}\}.$$

The following example shows that the above result is essentially better than that of Theorem 1.14 of [B1].

1.3. EXAMPLE. If $E = \bar{B}_n$, we have (see [B2]):

$$D_{v+} V_E(x) = (1 - x^2 + (x \cdot v)^2)^{1/2} (1 - x^2)^{-1/2} = (1 - G(x, v))^{1/2} (1 - x^2)^{-1/2},$$

for $|x| < 1$. Hence, if $n > 1$,

$$\begin{aligned} |\operatorname{grad}_+ V_E(x)| &= (1 - x^2)^{-1/2} < (n - 1 + (1 - x^2)^{-1})^{1/2} \\ &= |(D_{e_1+} V_E(x), \ldots, D_{e_n+} V_E(x))|. \end{aligned}$$

We want to extend Theorem 1.1 and Corollary 1.2 to the case of complex polynomials. For this we need the lemma below.

1.4. LEMMA. *Let $E \subset\subset \mathbf{R}^n$ and $\varphi\colon E \to \mathbf{R}_+$, $A\colon S^{n-1} \to \mathbf{R}_+$ be arbitrary functions and let $\omega\colon N_0 \to \mathbf{R}_+$ be a non-decreasing function. If for every $p \in \mathbf{R}[x_1, \ldots, x_n]$ and for each $x \in E$, $v \in S^{n-1}$,*

$$|D_v p(x)| \leqslant A(v) \, \omega(\deg p) \, \varphi(x) \, \|p\|_E, \tag{1.5}$$

then for each $p \in \mathbf{C}[x_1, \ldots, x_n]$ inequality (1.5) *holds for the same x and v.*

Proof. Let $p \in \mathbf{C}[x_1, ..., x_n]$ be fixed. We have: $p = q + ir$, where $q, r \in \mathbf{R}[x_1, ..., x_n]$ and $\deg p = \max(\deg q, \deg r)$. For a fixed $\tau \in S^1$ consider the polynomial $p_\tau = \tau_1 q + \tau_2 r$. It is obvious that $p_\tau \in \mathbf{R}[x_1, ..., x_n]$ and $\deg p_\tau \leqslant \deg p$. Moreover, by Schwarz's inequlity, we have

$$|p_\tau(x)| \leqslant |p(x)|$$

and

$$\|p_\tau\|_E \leqslant \|p\|_E.$$

Now, by (1.5), we obtain

$$\begin{aligned}|D_v p(x)| &= \sup\{|D_v p_\tau(x)| : \tau \in S^1\}\\ &\leqslant A(v)\,\omega(\deg p)\,\varphi(x)\sup\{\|p_\tau\|_E : \tau \in S^1\}\\ &\leqslant A(v)\,\omega(\deg p)\,\varphi(x)\,\|p\|_E.\end{aligned}$$

The proof is completed.

By Theorem 1.1 and Lemma 1.4 we easily obtain the following.

1.6. THEOREM. *Let E be a compact set E in $\mathbf{R}^n$ which is a closure of an open bounded subset. For every $x \in \operatorname{int}(E)$ we have the following inequality for a complex polynomial p with* $\deg p \leqslant k$,

$$|D_v p(x)| \leqslant k D_{v+} V_E(x)\,\|p\|_E,$$

for every $v \in S^{n-1}$ and

$$|\operatorname{grad} p(x)|_* \leqslant k\,|\operatorname{grad}_+ V_E(x)|\,\|p\|_E.$$

Here the norm $|\cdot|_*$ in $\mathbf{C}^n$ *is given by Turowicz's formula (see* [D])

$$|z|_* := \sup\{|z \cdot w| : w \in S^{n-1}\} = \left(\frac{|z|^2 + |z^2|}{2}\right)^{1/2}, \qquad z \in \mathbf{C}^n,$$

where $z^2 = z_1^2 + \cdots + z_n^2$.

1.7. *Remark.* We have noted in [B1] that Theorem 1.1 may be thought of as a generalization of Bernstein's inequality. Analogously, Theorem 1.6 extends the classical Bernstein–Markov inequality (see the next section). It follows from the Example 1.3 that the second inequality in Theorem 1.6 may be false if we replace Turowicz's norm by Euclidean norm in $\mathbf{C}^n$. Namely, let E be the unit (closed) Euclidean ball in $\mathbf{R}^2$. Consider a polynomial p of the form

$$p(x_1, x_2) = x_1 + ix_2.$$

Then $\deg p = 1$, $\|p\|_E = 1$, $\operatorname{grad} p(x_1, x_2) = (1, i)$, $|\operatorname{grad} p(x_1, x_2)|_* = 1$ and $|\operatorname{grad} p(x_1, x_2)| = \sqrt{2}$. However, by Example 1.3, $|\operatorname{grad}_+ V_E(0)| = 1$ and the inequality

$$|\operatorname{grad} p(0)| \leqslant |\operatorname{grad}_+ V_E(0)| \; \|p\|_E$$

is impossible.

2. Extension of Classical Inequalities for Polynomials from the Interval $[-1, 1]$ to the Case of a Convex Symmetric Subset of $\mathbf{R}^n$

In this section we extend the following three classical inequalities:

(a) (Bernstein, 1937)

$$|p'(x)| \leqslant k(1 - x^2)^{-1/2} \left(\|p\|^2_{[-1,1]} - p^2(x)\right)^{1/2},$$

where $x \in (-1, 1)$ and p is a real polynomial of one variable with $\deg p \leqslant k$;

(b) (Markov, 1889; Bernstein 1912)

$$|p'(x)| \leqslant k(1 - x^2)^{-1/2} \|p\|_{[-1,1]},$$

where $x \in (-1, 1)$, p is a complex polynomial of one variable with $\deg p \leqslant k$;

(c) (Markov, 1889)

$$\|p'\|_{[-1,1]} \leqslant k^2 \|p\|_{[-1,1]},$$

where p is a complex polynomial of one variable with $\deg p \leqslant k$.

Let E be a compact, convex and symmetric (with respect to the origin) subset of $\mathbf{R}^n$ with nonempty interior. By E^* we denote its polar

$$E^* := \{ y \in \mathbf{R}^n : x \cdot y \leqslant 1 \text{ for each } x \in E \},$$

where "$\cdot$" denotes the scalar product in $\mathbf{R}^n$. Define

$$f(x) := \sup\{ |x \cdot y| : y \in E^* \}, \qquad \text{for} \quad x \in \mathbf{R}^n,$$

and

$$f^*(x) := \sup\{ |x \cdot y| : y \in E \}, \qquad \text{for} \quad x \in \mathbf{R}^n.$$

Then both functions f and f^* are norms in $\mathbf{R}^n$ and E, E^* are unit balls for

f and f^*, respectively. There exists the following canonical isomorphism between the space $\mathbf{R}^n$ and its dual $(\mathbf{R}^n)^* = \mathscr{L}(\mathbf{R}^n; \mathbf{R})$

$$l\colon \mathbf{R}^n \ni x \to \{y \to x \cdot y\} \in (\mathbf{R}^n)^*.$$

Since $f^* \circ l^{-1}$ is a norm in $(\mathbf{R}^n)^*$ induced by the norm f, we can briefly say that f^* is a dual norm in $\mathbf{R}^n$ and E^* is a dual ball in $\mathbf{R}^n$. So, without loss of generality, if we consider a compact, convex and symmetric subset of $\mathbf{R}^n$ with nonempty interior, then one can assume that it is given a norm f (E is a unit ball with respect to this norm) and dual ball E^*. We formulate our main result in a language of norms. The following theorem is a nice application of Theorems 1.1 and 1.6 and Corollary 1.2 and generalizes the classical results for the interval $[-1, 1]$.

2.1. THEOREM. *Let f be a fixed norm in $\mathbf{R}^n$. Put $E = \{f(x) \leqslant 1\}$. Then we have*

(a) (*Bernstein inequality*) *If $p \in \mathbf{R}[x_1, ..., x_n]$ $(\deg p \leqslant k)$, then*

$$|D_v p(x)| \leqslant k f(v)(1 - f^2(x))^{-1/2} (\|p\|_E^2 - p^2(x))^{1/2},$$

for $f(x) < 1$.

(b) (*Bernstein–Markov inequality*) *If $p \in \mathbf{C}[x_1, ..., x_n]$ $(\deg p \leqslant k)$, then*

$$|D_v p(x)| \leqslant k f(v)(1 - f^2(x))^{-1/2} \|p\|_E,$$

for $f(x) < 1$.

(c) (*Markov inequality*) *If $p \in \mathbf{C}[x_1, ..., x_n]$ $(\deg p \leqslant k)$, then*

$$\|D_v p\|_E \leqslant f(v)\, k^2 \|p\|_E.$$

(c′) *If $p \in \mathbf{C}[x_1, ..., x_n]$ $(\deg p \leqslant k)$, then*

$$|\operatorname{grad} p(x)|_* \leqslant \frac{1}{2} \operatorname{diam}(E^*)\, k^2 \|p\|_E = \frac{1}{2c(E)} k^2 \|p\|_E,$$

if $f(x) < 1$.

*Here $c(E)$ denotes the L-*capacity *of the set E,*

$$c(E) := \lim \inf_{|z| \to \infty} |z| \exp(-V_E^*(z)),$$

(*cf.* [S]), *which is a generalization of logarithmic capacity (equivalently transfinite diameter) in the case $n = 1$ and plays an important role in the pluripotential theory (cf.* [S, BT, KO, KL]). *Here V_E^* stands for the upper regularization of V_E.*

Proof. Part (a) was proved in [B1] in the case where $v = e_j$, $j = 1, ..., n$. In the general case the proof is similar and we omit it. Using Lemma 1.4 and part (a), we obtain part (b). In order to prove part (c), fix $x \in E$ with $f(x) = 1$. Consider the polynomial $Q(t) = D_v p(tx) \in \mathbf{C}[t]$. It is clear that $\deg Q \leqslant k - 1$. If $t \in \mathbf{R}$, $|t| < 1$, we have

$$|Q(t)| \leqslant k(1 - t^2)^{-1/2} f(v) \, \|p\|_E.$$

By a standard argument (see, e.g., [CH]), we obtain now

$$|Q(t)| \leqslant k^2 f(v) \, \|p\|_E, \qquad |t| \leqslant 1.$$

The least inequality easily implies inequality (c).

The left-hand side inequality of (c′) is an easy corollary to (c), while the right-hand side equality is a consequence of the formula

$$c(E) = \frac{1}{\operatorname{diam}(E^*)},$$

which follows from the representation

$$V_E(z) = \log h(\sup\{\tfrac{1}{2}\,|z \cdot w + 1| + \tfrac{1}{2}\,|z \cdot w - 1| : w \in \operatorname{extr} E^*\}),$$

where $h(t) = t + \sqrt{t^2 - 1}$, for $t \geqslant 1$ (see, e.g., [B1]).

2.2. *Remark.* It is easy to see that the constants in the above theorem are the best possible. In the special case of $E = \bar{B}_n$ inequality (c′) for real polynomials was obtained by Kellog [KE] in 1927. A simple proof of Kellog's result has been recently obtained by Jonsson [J].

One can easily verify that the left-hand side inequality of (c′) for real polynomials is equivalent to satisfying Wilhemsen's conjecture (cf. [W]), which was formulated in 1974. However, we do not believe that this conjecture is true for any convex body in $\mathbf{R}^n$. Probably Wilhemsen's conjecture is false for the standard simplex in $\mathbf{R}^n$.

3. Geometrical Approach to the Bernstein and Markov Property

Let $E \subset\subset \mathbf{R}^n$. If $\alpha > 0$ and a field $\mathbf{K}$ ($\mathbf{K} = \mathbf{R}$ *or* $\mathbf{K} = \mathbf{C}$) is fixed, then by $G_{E,x}^{(\alpha)} = G_x^{(\alpha)}$ we shall denote the following set of gradients of polynomials at a fixed point $x \in E$:

$$G_x^{(\alpha)} = \left\{ \frac{1}{(\deg p)^\alpha} \operatorname{grad} p(x) : p \in \mathbf{K}[x_1, ..., x_n], \deg p \geqslant 1, \|p\|_E \leqslant 1 \right\}.$$

It is clear that $G_x^{(\alpha)}$ is symmetric (with respect to the origin) and has nonempty interior. Moreover, if $x \in \operatorname{int}(E)$, then the set $G_x^{(1)}$ is bounded (and so is for $\alpha > 1$). It is obvious that

$$\operatorname{diam}(G_x^{(\alpha)}) = \operatorname{diam}(\operatorname{conv}(G_x^{(\alpha)}).$$

We can now state a theorem which yields a geometrical interpretation of results of [B1]. Its proofs can be easily derived from [B1, B2] and from the results of the present paper. We consider the case where $\mathbf{K} = \mathbf{R}$.

3.1. THEOREM. *Let E be a compact fat set in $\mathbf{R}^n$.*

(a) *For every $x \in \operatorname{int}(E)$ we have the following inequality*:

$$\operatorname{diam}(G_x^{(1)}) \leqslant 2 \sup\{D_{v+} V_E(x) : v \in S^{n-1}\}.$$

(b) *For almost every $x \in \operatorname{int}(E)$ (with respect to the Lebesgue measure) the following inequality holds*:

$$\operatorname{vol}(\operatorname{conv}(G_x^{(1)})) \leqslant \frac{1}{n!} \lambda(x).$$

Here $\lambda(x)$ denotes the density of the complex equilibrium measure λ_E *associated with the compact set E (cf. Theorem* 1.15 *of* [B1]).

We conjecture that in both inequalities in Theorem 3.1 we have the equality for almost every $x \in \operatorname{int}(E)$.

From the statement (b) of Theorem 3.1 one can deduce that the function $\varphi(x) = \operatorname{diam}(G_x^{(1)})$ is bounded from above by a function from $L^1(E)$ (cf. [B1]).

In the special case of a ball in $\mathbf{R}^n$ we have a slightly surprising theorem.

3.2. THEOREM. *Let f be a fixed norm in $\mathbf{R}^n$. Put $E = \{f(x) \leqslant 1\}$. We have the inclusion*

$$G_x^{(1)} \subset (1 - f^2(x))^{-1/2} E^*$$

for $f(x) < 1$ *with equality at $x = 0$, and the Markov equality*

$$G_x^{(2)} = E^*,$$

for $x \in E$. Moreover, we have

$$G_x^{(\alpha)} \subset (1 - f^2(x))^{-(1/2)(2-\alpha)} E^*, \qquad \textit{for} \quad x \in \operatorname{int}(E),$$

where $1 < \alpha < 2$.

Proof. The inclusion $G_x^{(1)} \subset (1-f^2(x))^{-1/2} E^*$ is an equivalent form of Theorem 2.1(b) for real polynomials. In order to prove the equality observe that inclusion $\subset$ is an equivalent form of Theorem 2.1(c) fo real polynomials. The inclusion $\supset$ is a consequence of fact that the real polynomial $p(x) = w \cdot x$, where $w \in E^*$ is fixed, satisfies conditions: $\|p\|_E \leqslant 1$ and grad $p = w$. Combining the first inclusion with the Markov equality and the elementary inequality

$$\min(a, b) \leqslant a^t b^{1-t},$$

where $a, b \geqslant 0$, $0 \leqslant t \leqslant 1$, we obtain the second inclusion.

Theorem 3.2 becomes more interesting in comparison to the following facts.

The proposition below seems to be known. Probably, a proof of this proposition can be found in [SM].

3.3. Proposition. *Let (X, f) be a real normed space and (X^*, f^*) its dual. Denote by B and B^* the unit balls for f and f^*, respectively. If the norm f is Gâteaux (Fréchet) differentiable at some point $x \in X$, then $d_x f \in \operatorname{extr} B^*$. Moreover, $d_x f$ is a point of strict convexity of B^*.*

It is well known that every continuous convex function defined on a finite dimensional space X is almost everywhere Fréchet differentiable. Combining this fact with Proposition 3.3 and the well-known Straszewicz theorem, one can obtain that if f is a norm in $\mathbf{R}^n$, E denotes its closed unit ball and $\mathscr{D} = \{x \in \mathbf{R}^n : f \text{ is Fréchet differentiable at } x\}$, then

$$E^* = \operatorname{conv}\overline{\{\operatorname{grad} f(x) : x \in \mathscr{D}\}}.$$

Tis yields the following sharp version of Markov's inequality.

3.4. Corollary. *Let f be a fixed norm in $\mathbf{R}^n$. Put $E = \{f(x) \leqslant 1\}$. Then we have*

$$G_x^{(2)} = \operatorname{conv}\overline{\{\operatorname{grad} f(x) : x \in \mathscr{D}\}}$$

for $x \in E$.

Finally, consider the case where $\mathbf{K} = \mathbf{C}$ and E is the unit ball for a norm f in $\mathbf{R}^n$. Denote by $\mathbf{C}(E^*)$ the complexification of the dual ball E^* given by

$$\mathbf{C}(E^*) := \{z \in \mathbf{C}^n : |z \cdot w| \leqslant 1 \text{ for every } w \in E\}.$$

Note that such a complexification was considered in [B3] in connection with a problem of complex foliation of $\mathbf{C}^n \setminus E$.

If $\mathbf{K} = \mathbf{C}$, it is easy to see that we have

$$G_x^{(2)} = \mathbf{C}(E^*).$$

In particular, if $E = \bar{B}_n = E^*$ we have (cf. [D])

$$\mathbf{C}(\bar{B}_n) = \{z \in \mathbf{C}^n : |z|^2 + |z^2| \leqslant 2\},$$

and one can prove that (for $n > 1$)

$$\mathbf{C}(\bar{B}_n) = \operatorname{conv}\{z = x + iy \in \mathbf{C}^n : |x| = |y| = 1 \text{ and } x \cdot y = 0\}.$$

References

[B1] M. Baran, Bernstein type theorems for compact sets in $\mathbf{R}^n$, *J Approx. Theory* **69** (1992), 156–166.

[B2] M. Baran, Siciak's extremal function and complex equilibrium measure for compact sets in $\mathbf{R}^n$, *Proc. Amer. Math. Soc.*, to appear.

[B3] M. Baran, Plurisubharmonic extremal function and complex foliation for a complement of a convex subset of $\mathbf{R}^n$, *Michigan Math. J.* **32** (1992), 395–404.

[BT] E. Bedford and B. A. Taylor, A new capacity for plurisubharmonic functions, *Acta Math.* **149** (1982), 1–40.

[CH] E. W. Cheney, "Introduction to Approximation Theory," New York, 1966.

[D] L. M. Drużkowski, Effective formula for the crossnorm in complexified unitary spaces, *Univ. Iagiel. Acta Math.* **16** (1974), 47–53.

[J] A. Jonsson, On Markov's and Bernstein's inequalities in the unit ball, preprint.

[KE] O. D. Kellog, On bounded polynomials in several variables, *Math. Z.* **27** (1927), 55–64.

[KL] M. Klimek, "Pluripotential Theory," Oxford Univ. Press, Oxford, 1991.

[KO] S. Kołodziej, The logarithmic capacity in $\mathbf{C}^n$, *Ann. Polon. Math.* **48** (1988), 253–267.

[S] J. Siciak, Extremal plurisubharmonic functions in $\mathbf{C}^n$, *Ann. Polon. Math.* **39** (1981), 175–211.

[SM] V. M. Shmul'yan, Sur la dérivabilité de la norme dans l'espace de Banach, *Dokl. Akad. Nauk. SSSR (N.S.)* **27** (1940), 643–648.

[W] D. R. Wilhelmsen, A Markov inequality in several dimensions, *J. Approx. Theory* **11** (1974), 216–220.